Groupe 03 Projet DANJOU - DUROUSSEAU

Contents

Exercise 1 : Negative weighted mixture

Definition

Question 1 The conditions for a function f to be a probability density are :

- f is defined on $\mathbb{R}$
- f is non-negative, ie $f(x) \geq 0$, $\forall x \in \mathbb{R}$
- f is Lebesgue-integrable
- and $\int_{\mathbb{R}} f(x)\, dx = 1$

The function f, to be a density, needs to be non-negative, ie $f(|x|) \geq 0$ when $|x| \to \infty$

We have, when $|x| \to \infty$, $f_i(|x|) \sim exp(\frac{-x^2}{\sigma_i^2})$ for $i = 1, 2$

Then, $f(|x|) \sim exp(\frac{-x^2}{\sigma_1^2}) - exp(\frac{-x^2}{\sigma_2^2})$

We want $f(|x|) \geq 0$, ie, $exp(\frac{-x^2}{\sigma_1^2}) - exp(\frac{-x^2}{\sigma_2^2}) \geq 0$

$\Leftrightarrow \sigma_1^2 \geq \sigma_2^2$

We can see that f_1 dominates the tail behavior.

Question 2 For given parameters (μ_1, σ_1^2) and (μ_2, σ_2^2), we have $\forall x \in \mathbb{R}$, $f(x) \geq 0$

$\Leftrightarrow \frac{1}{\sigma_1} exp(\frac{-(x-\mu_1)^2}{2\sigma_1^2}) \geq \frac{a}{\sigma_2} exp(\frac{-(x-\mu_2)^2}{2\sigma_2^2})$

$\Leftrightarrow 0 < a \leq a^* = \min_{x \in \mathbb{R}} \frac{f_1(x)}{f_2(x)} = \min_{x \in \mathbb{R}} \frac{\sigma_2}{\sigma_1} exp(\frac{(x-\mu_2)^2}{2\sigma_2^2} - \frac{(x-\mu_1)^2}{2\sigma_1^2})$

To find a^*, we just have to minimize $g(x) := \frac{(x-\mu_2)^2}{2\sigma_2^2} - \frac{(x-\mu_1)^2}{2\sigma_1^2}$

First we derive g: $\forall x \in \mathbb{R}, g'(x) = \frac{x-\mu_2}{\sigma_2^2} - \frac{x-\mu_1}{\sigma_1^2}$

We search x^* such that $g'(x^*) = 0$

$\Leftrightarrow x^* = \frac{\mu_2\sigma_1^2 - \mu_1\sigma_2^2}{\sigma_1^2 - \sigma_2^2}$

Then, we compute $a^* = \frac{f_1(x^*)}{f_2(x^*)}$

We call $C \in \mathbb{R}$ the normalization constant such that $f(x) = C(f_1(x) - af_2(x))$

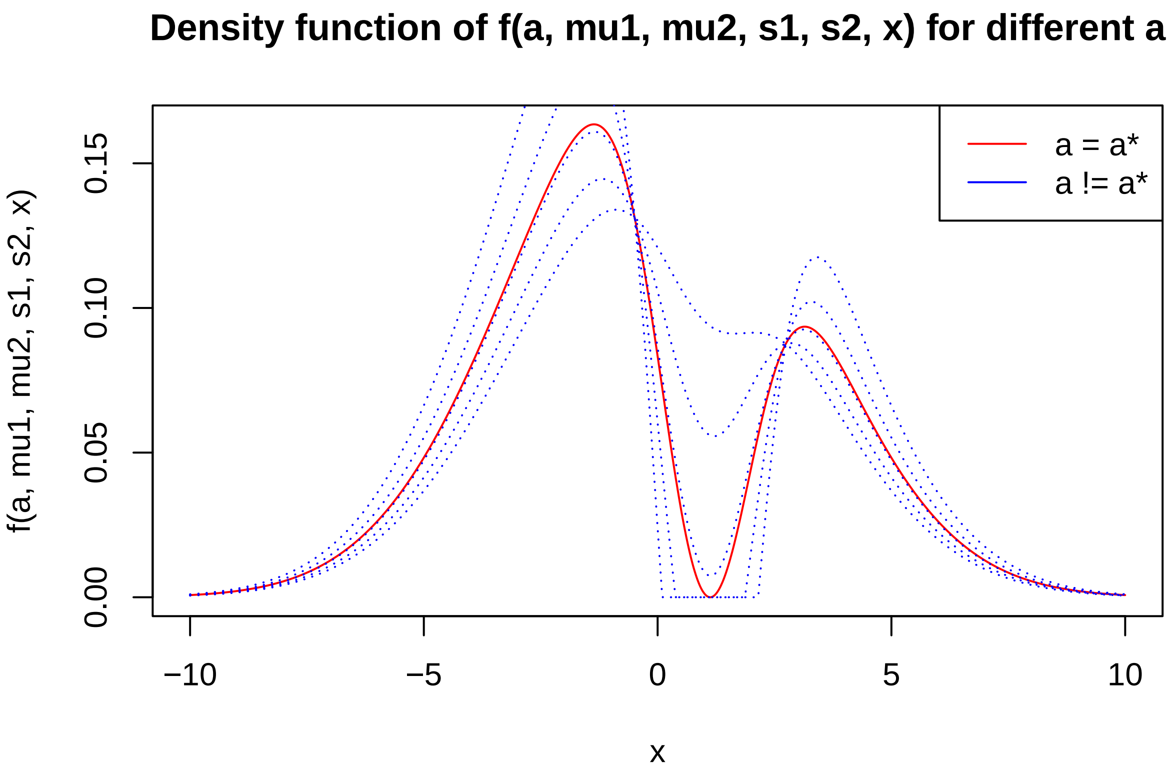
To find C, we know that $1 = \int_{\mathbb{R}} f(x)\, dx = \int_{\mathbb{R}} C(f_1(x) - af_2(x))\, dx = C \int_{\mathbb{R}} f_1(x)\, dx - Ca \int_{\mathbb{R}} f_2(x)\, dx = C(1-a)$ as f_1 and f_2 are density functions and by linearity of the integrals.

$\Leftrightarrow C = \frac{1}{1-a}$

```r
f <- function(a, mu1, mu2, s1, s2, x) {
  fx <- dnorm(x, mu1, s1) - a * dnorm(x, mu2, s2)
  fx[fx < 0] <- 0
  return(fx / (1 - a))
}
```

```r
a_star <- function(mu1, mu2, s1, s2) {
  x_star <- (mu2 * s1^2 - mu1 * s2^2) / (s1^2 - s2^2)
  return(dnorm(x_star, mu1, s1) / dnorm(x_star, mu2, s2))
}
```

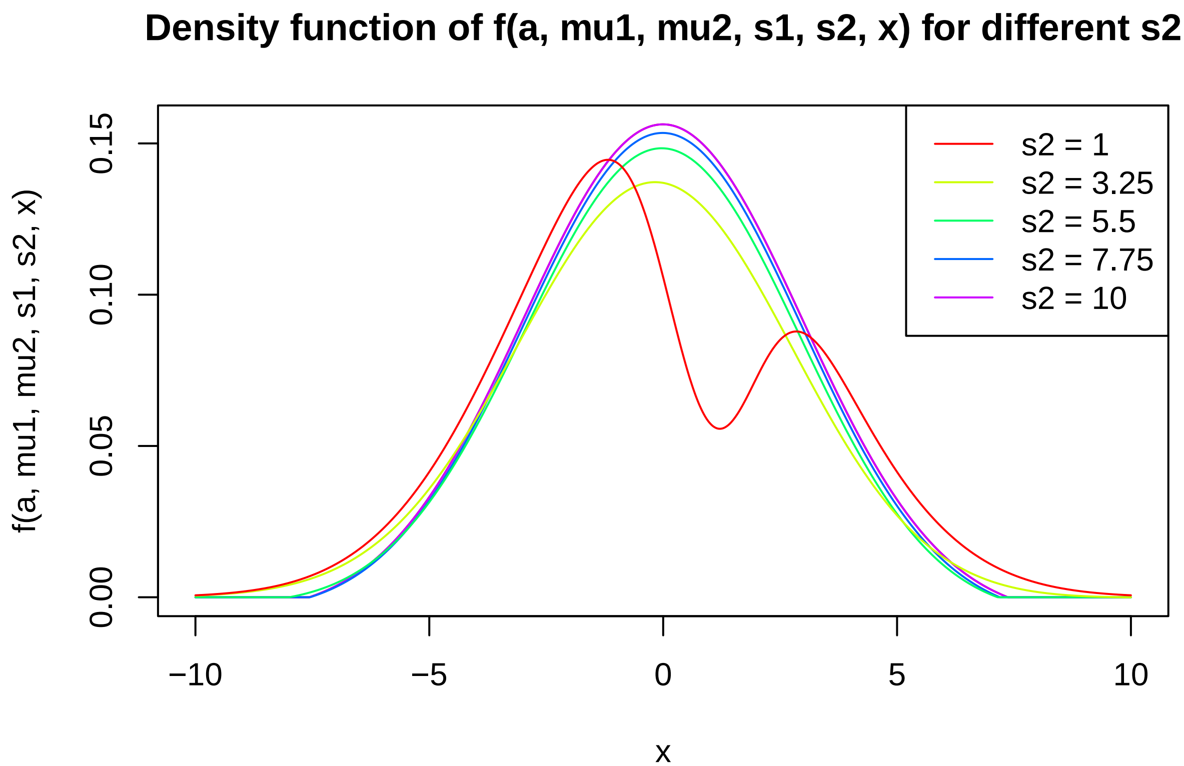
```r
mu1 <- 0
mu2 <- 1
s1 <- 3
s2 <- 1

x <- seq(-10, 10, length.out = 1000)
as <- a_star(mu1, mu2, s1, s2)
a_values <- c(0.1, 0.2, 0.3, 0.4, 0.5, as)

plot(x, f(as, mu1, mu2, s1, s2, x),
     type = "l",
     col = "red",
     xlab = "x",
     ylab = "f(a, mu1, mu2, s1, s2, x)",
     main = "Density function of f(a, mu1, mu2, s1, s2, x) for different a"
)
for (i in (length(a_values) - 1):1) {
  lines(x, f(a_values[i], mu1, mu2, s1, s2, x), lty = 3, col = "blue")
}
legend("topright", legend = c("a = a*", "a != a*"), col = c("red", "blue"), lty = 1)
```



Question 3

We observe that for small values of a, the density f is close to the density of $f_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$. When a increases, the shape of evolves into the combinaison of two normal distributions. We observe that for $a = a^*$, the density the largest value of a for which the density is still a density function, indeed for $a > a^*$, the function f takes negative values so it is no longer a density.

```
s2_values <- seq(1, 10, length.out = 5)
a <- 0.2

plot(x, f(a, mu1, mu2, s1, max(s2_values), x),
     type = "l",
     xlab = "x",
     ylab = "f(a, mu1, mu2, s1, s2, x)",
     col = "red",
     main = "Density function of f(a, mu1, mu2, s1, s2, x) for different s2"
)

for (i in length(s2_values):1) {
  lines(x, f(a, mu1, mu2, s1, s2_values[i], x), lty = 1, col = rainbow(length(s2_values))[i])
}

legend("topright", legend = paste("s2 =", s2_values), col = rainbow(length(s2_values)), lty = 1)
```



We observe that when $\sigma_2^2 = 1$, the density f has two peaks and when $\sigma_2^1 > 1$, the density f has only one peak.

```
mu1 <- 0
mu2 <- 1
sigma1 <- 3
sigma2 <- 1
a <- 0.2
as <- a_star(mu1, mu2, sigma1, sigma2)

cat(sprintf("a* = %f, a = %f, a <= a* [%s]", as, a, a <= as))
```

```
## a* = 0.313138, a = 0.200000, a <= a* [TRUE]
```

We have $\sigma_1^2 \geq \sigma_2^2$ and $0 < a \leq a^*$, so the numerical values are compatible with the constraints defined above.

Inverse c.d.f Random Variable simulation

Question 4 To prove that the cumulative density function F associated with f is available in closed form, we need to compute $F(x) = \int_{-\infty}^{x} f(t)\,dt = \frac{1}{1-a}(\int_{-\infty}^{x} f_1(t)\,dt - a\int_{-\infty}^{x} f_2(t)\,dt) = \frac{1}{1-a}(F_1(x) - aF_2(x))$ where F_1 and F_2 are the cumulative density functions of $f_1 \sim \mathcal{N}(\mu 1, \sigma_1^2)$ and $f_2 \sim \mathcal{N}(\mu 2, \sigma_2^2)$ respectively.

Then, F is a closed-form as a finite sum of closed forms.

```
F <- function(a, mu1, mu2, s1, s2, x) {
  Fx <- pnorm(x, mu1, s1) - a * pnorm(x, mu2, s2)
  return(Fx / (1 - a))
}
```

To construct an algorithm that returns the value of the inverse function method as a function of $u \in (0, 1)$, of the parameters $a, \mu_1, \mu_2, \sigma_1, \sigma_2, x$, and of an approximation precision ϵ, we can use the bisection method.

We fixe $\epsilon > 0$.
We set $u \in (0, 1)$.
We define $L = -10$ and $U = -L$, the bounds and $M = \frac{L+U}{2}$, the middle of our interval.
While $U - L > \epsilon$:

- We compute $F(M) = \frac{1}{1-a}(F_1(M) - aF_2(M))$
- If $F(M) < u$, we set $L = M$
- Else, we set $U = M$
- We set $M = \frac{L+U}{2}$

End while

For the generation of random variables from F, we can use the inverse transform sampling method.

We set X as an empty array and n the number of random variables we want to generate.
We fixe $\epsilon > 0$.
For $i = 1, \ldots, n$:

- We set $u \in (0, 1)$.

- We define $L = -10$ and $U = -L$, the bounds and $M = \frac{L+U}{2}$, the middle of our interval.

- While $U - L > \epsilon$:

- We compute $F(M) = \frac{1}{1-a}(F_1(M) - aF_2(M))$

- If $F(M) < u$, we set $L = M$

- Else, we set $U = M$

- We set $M = \frac{L+U}{2}$

- End while

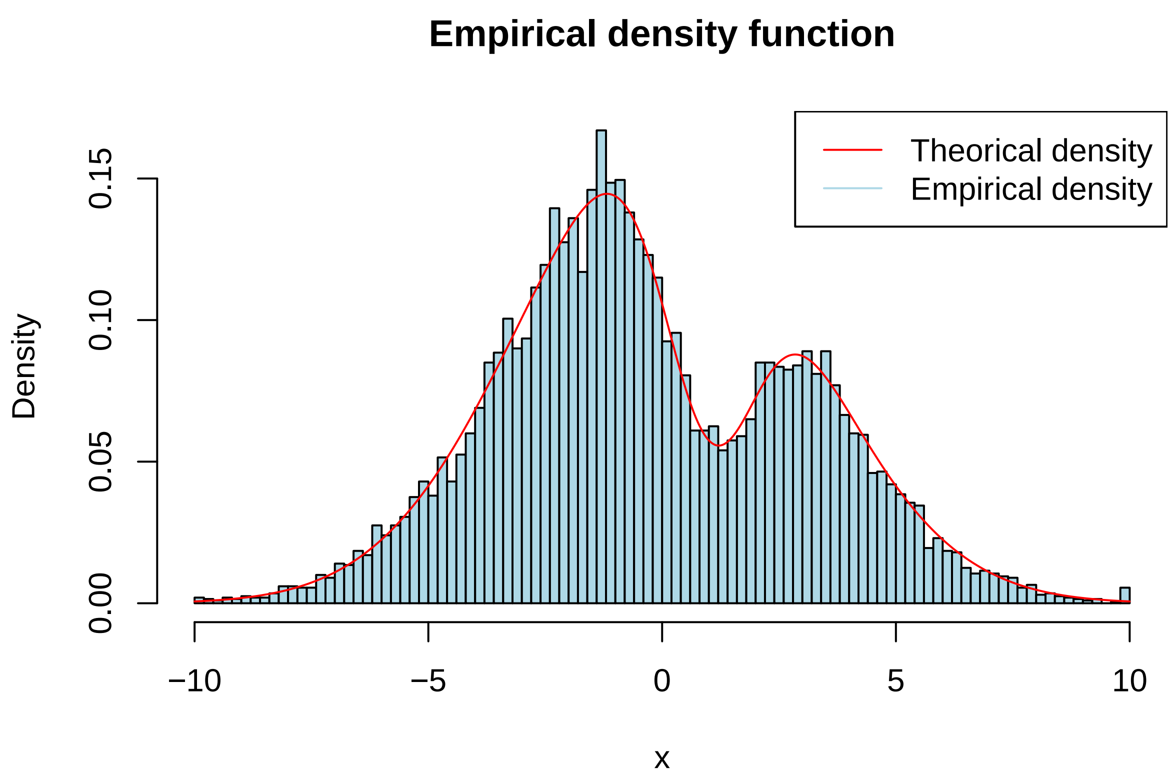
- We add M to X

End for We return X

```
inv_cdf <- function(n) {
  X <- numeric(n)
  for (i in 1:n) {
    u <- runif(1)
    L <- -10
```

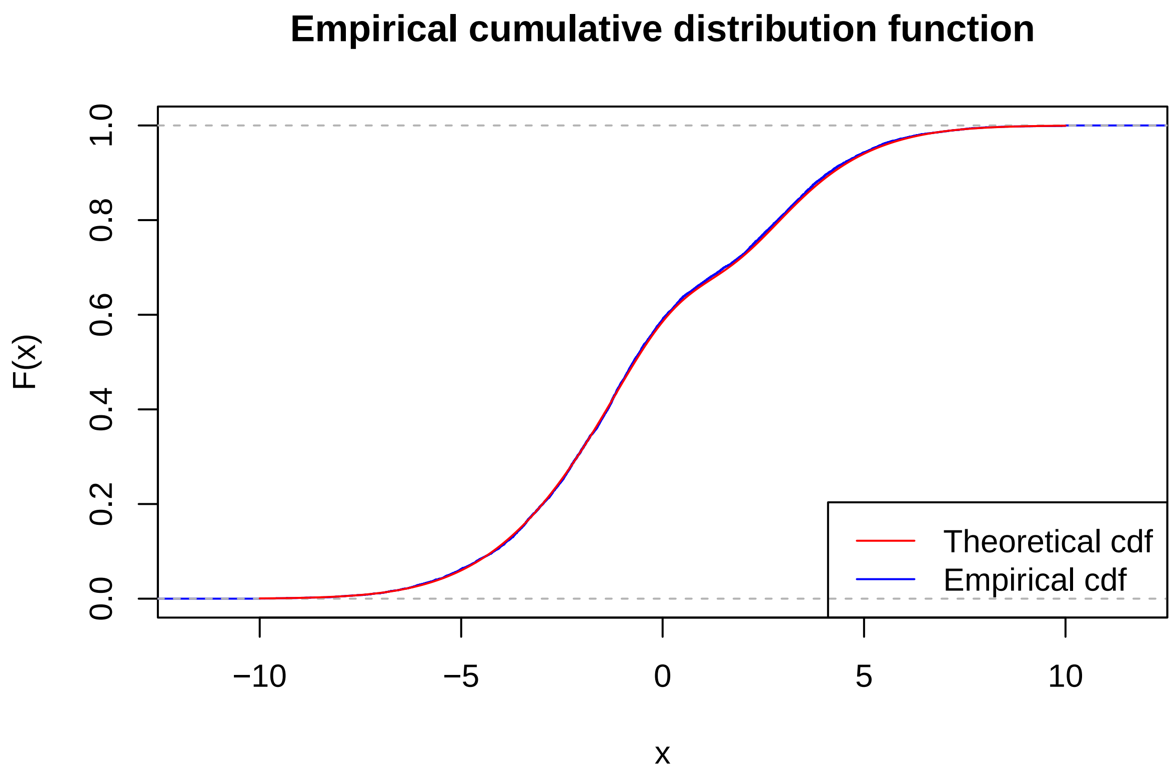
```
    U <- -L
    M <- (L + U) / 2
    while (U - L > 1e-6) {
      FM <- F(a, mu1, mu2, s1, s2, M)
      if (FM < u) {
        L <- M
      } else {
        U <- M
      }
      M <- (L + U) / 2
    }
    X[i] <- M
  }
  return(X)
}
```

```
set.seed(123)
n <- 10000
X <- inv_cdf(n)
x <- seq(-10, 10, length.out = 1000)

hist(X, breaks = 100, freq = FALSE, col = "lightblue", main = "Empirical density function", xlab = "x")
lines(x, f(a, mu1, mu2, s1, s2, x), col = "red")
legend("topright", legend = c("Theorical density", "Empirical density"), col = c("red", "lightblue"), l
```



Question 5

```
plot(ecdf(X), col = "blue", main = "Empirical cumulative distribution function", xlab = "x", ylab = "F(
lines(x, F(a, mu1, mu2, s1, s2, x), col = "red")
legend("bottomright", legend = c("Theoretical cdf", "Empirical cdf"), col = c("red", "blue"), lty = 1)
```



We can see of both graphs that the empirical cumulative distribution function is close to the theoretical one.

Accept-Reject Random Variable simulation

Question 6 To simulate under f using the accept-reject algorithm, we need to find a density function g such that $f(x) \leq Mg(x)$ for all $x \in \mathbb{R}$, where M is a constant.

Then, we generate $X \sim g$ and $U \sim \mathcal{U}([0,1])$.
We accept $Y = X$ if $U \leq \frac{f(X)}{Mg(X)}$. Return to 1 otherwise.

The probability of acceptance is $\int_{\mathbb{R}} \frac{f(x)}{Mg(x)} g(x)\, dx = \frac{1}{M} \int_{\mathbb{R}} f(x)\, dx = \frac{1}{M}$

Here we pose $g = f_1$.

Then we have $\frac{f(x)}{g(x)} = \frac{1}{1-a}(1 - a\frac{f_2(x)}{f_1(x)})$

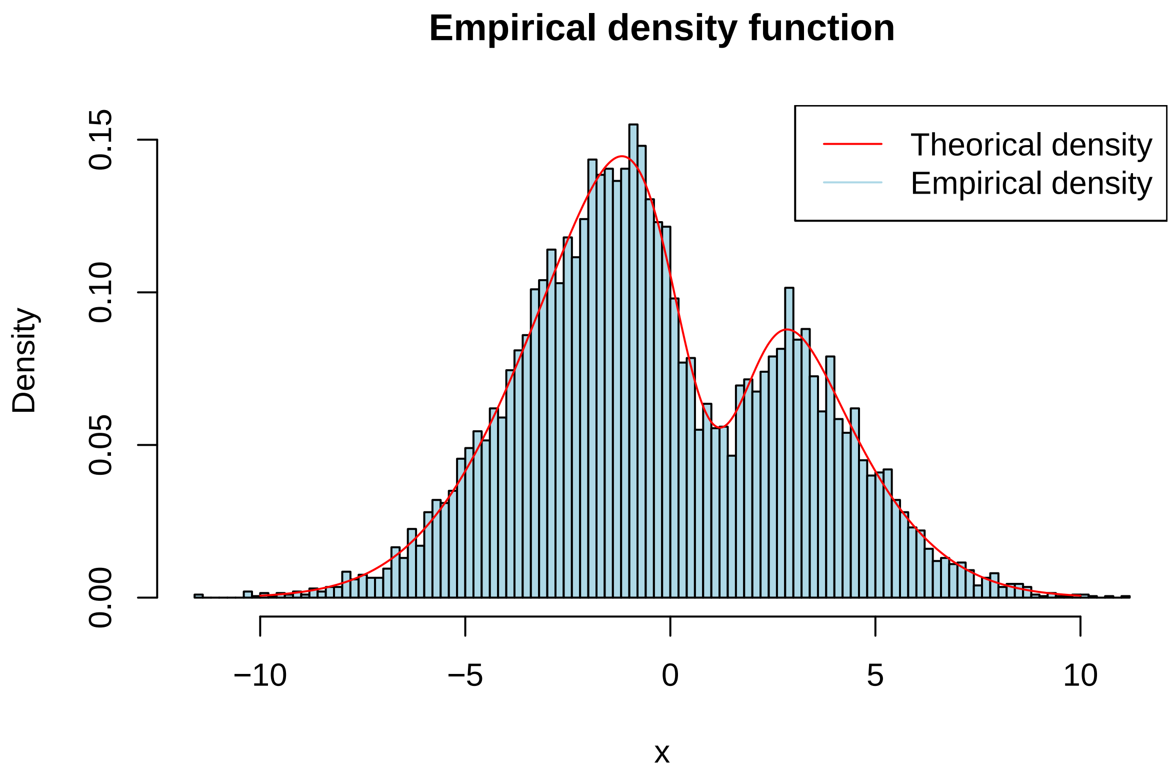
We pose earlier that $a^* = \min_{x\in\mathbb{R}} \frac{f_1(x)}{f_2(x)} \Rightarrow \frac{1}{a^*} = \max_{x\in\mathbb{R}} \frac{f_2(x)}{f_1(x)}$.

We compute in our fist equation : $\frac{1}{1-a}(1 - a\frac{f_2(x)}{f_1(x)}) \leq \frac{1}{1-a}(1 - \frac{a}{a^*}) \leq \frac{1}{1-a}$ because $a \leq a^* \Rightarrow 1 - \frac{a}{a^*} \leq 0$

To conclude, we have $M = \frac{1}{1-a}$ and the probability of acceptance is $\frac{1}{M} = 1 - a$

```r
accept_reject <- function(n, a) {
  X <- numeric(0)
  M <- 1 / (1 - a)

  while (length(X) < n) {
    Y <- rnorm(1, mu1, s1)
    U <- runif(1)

    if (U <= f(a, mu1, mu2, s1, s2, Y) / (M * dnorm(Y, mu1, s1))) {
      X <- append(X, Y)
    }
  }
  return(X)
}
```

```r
set.seed(123)
n <- 10000
a <- 0.2
X <- accept_reject(n, a)
x <- seq(-10, 10, length.out = 1000)

hist(X, breaks = 100, freq = FALSE, col = "lightblue", main = "Empirical density function", xlab = "x")
lines(x, f(a, mu1, mu2, s1, s2, x), col = "red")
legend("topright", legend = c("Theorical density", "Empirical density"), col = c("red", "lightblue"), l
```



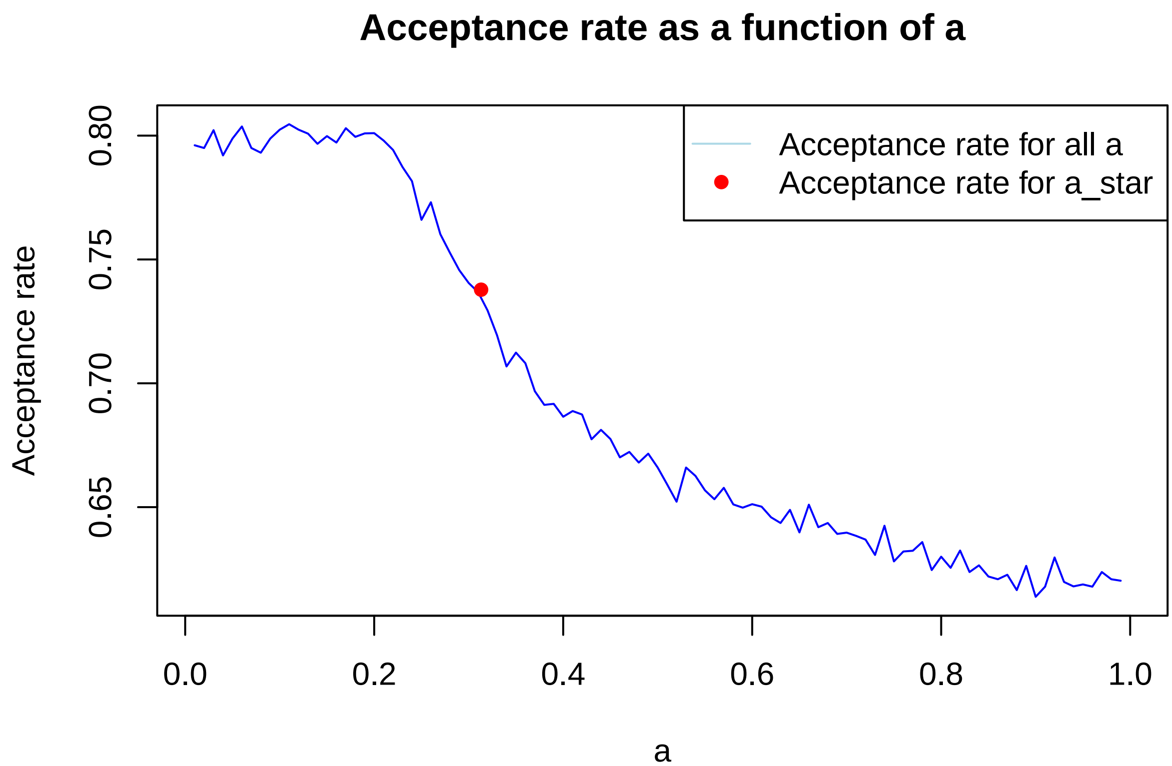
Question 7

```
set.seed(123)

acceptance_rate <- function(n, a = 0.2) {
  Y <- rnorm(n, mu1, s1)
  U <- runif(n)
  return(mean(U <= f(a, mu1, mu2, s1, s2, Y) / (M * dnorm(Y, mu1, s1))))
}

M <- 1 / (1 - a)
n <- 10000
cat(sprintf("[M = %.2f] Empirical acceptance rate: %f, Theoretical acceptance rate: %f \n", M, acceptan
```

```
## [M = 1.25] Empirical acceptance rate: 0.802600, Theoretical acceptance rate: 0.800000
```

```
set.seed(123)
a_values <- seq(0.01, 1, length.out = 100)
acceptance_rates <- numeric(length(a_values))
as <- a_star(mu1, mu2, s1, s2)

for (i in seq_along(a_values)) {
  acceptance_rates[i] <- acceptance_rate(n, a_values[i])
}
```

```
plot(a_values, acceptance_rates, type = "l", col = "blue", xlab = "a", ylab = "Acceptance rate", main =
points(as, acceptance_rate(n, as), col = "red", pch = 16)
legend("topright", legend = c("Acceptance rate for all a", "Acceptance rate for a_star"), col = c("ligh
```



Question 8

Random Variable simulation with stratification

Question 9 We consider a partition $\mathcal{P} = (D_0, D_1, ..., D_k)$, $k \in \mathbb{N}$ of $\mathbb{R}$ such that D_0 covers the tails of f_1 and f_1 is upper bounded and f_2 lower bounded in $D_1, \ldots, D_k$.

To simulate under f using the accept-reject algorithm, we need to find a density function g such that $f(x) \leq Mg(x)$ for all $x \in \mathbb{R}$, where M is a constant.

We generate $X \sim g$ and $U \sim \mathcal{U}([0,1])$ *(1)*.
We accept $Y = X$ if $U \leq \frac{f(X)}{Mg(X)}$. Otherwise, we return to *(1)*.

Here we pose $g(x) = f_1(x)$ if $x \in D_0$ and $g(x) = sup_{D_i} f_1(x)$ if $x \in D_i, i \in \{1, \ldots, k\}$.

To conclude, we have $M = \frac{1}{1-a}$ and the probability of acceptance is $r = \frac{1}{M} = 1 - a$

Question 10 Let $P_n = (D_0, \ldots D_n)$ a partition of $\mathbb{R}$ for $n \in \mathbb{N}$. We have $\forall x \in P_n$ and $\forall i \in \{0, \ldots, n\}$, $\lim_{n\to\infty} sup_{D_i} f_1(x) = f_1(x)$ and $\lim_{x\to\infty} inf_{D_i} f_2(x) = f_2(x)$.

$\Rightarrow \lim_{x\to\infty} g(x) = f(x)$

$\Rightarrow \lim_{x\to\infty} \frac{g(x)}{f(x)} = 1$ as $f(x) > 0$ as f is a density function.

$\Rightarrow \forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N}$ such that $\forall n \geq n_\epsilon$, $M = sup_{x \in P_n} \frac{g(x)}{f(x)} < \epsilon$

$\Rightarrow r = \frac{1}{M} > \frac{1}{\epsilon} := \delta \in]0, 1]$ where r is the acceptance rate defined in the question 10.

Question 11 We recall the parameters and the functions of the problem.

```
mu1 <- 0
mu2 <- 1
s1 <- 3
s2 <- 1
a <- 0.2

f1 <- function(x) {
  dnorm(x, mu1, s1)
}

f2 <- function(x) {
  dnorm(x, mu2, s2)
}

f <- function(x) {
  fx <- f1(x) - a * f2(x)
  fx[fx < 0] <- 0
  return(fx / (1 - a))
}

f1_bounds <- c(mu1 - 3 * s1, mu1 + 3 * s1)
```

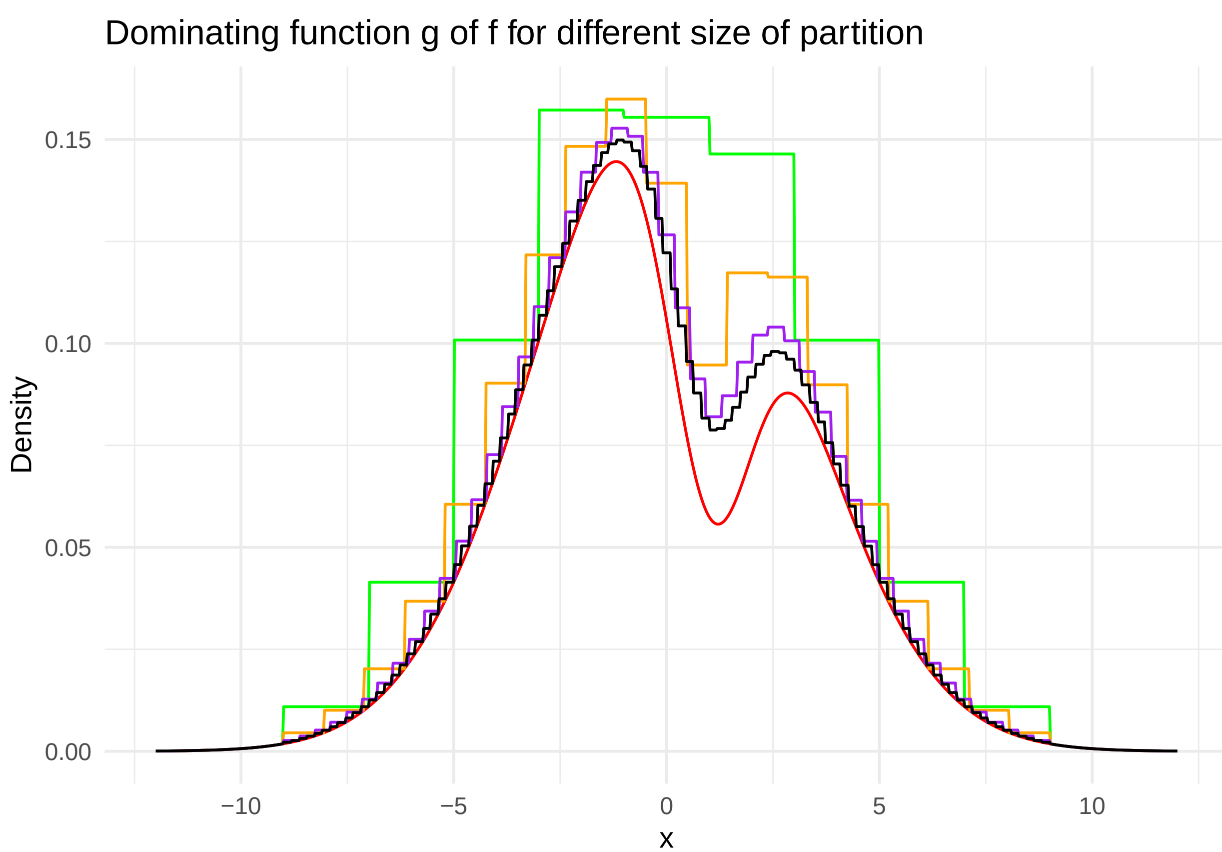
We implement the partition, the given g function to understand the behavior of g compared to f and the computation of the supremum and infimum of f_1 and f_2 on each partition.

```
create_partition <- function(k = 10) {
  return(seq(f1_bounds[1], f1_bounds[2], , length.out = k))
}

sup_inf <- function(f, P, i) {
  x <- seq(P[i], P[i + 1], length.out = 1000)
  f_values <- sapply(x, f)
  return(c(max(f_values), min(f_values)))
}

g <- function(X, P) {
  values <- numeric(0)
  for (x in X) {
    if (x <= P[1] | x >= P[length(P)]) {
      values <- c(values, 1 / (1 - a) * f1(x))
    } else {
      for (i in 1:(length(P) - 1)) {
        if (x >= P[i] & x <= P[i + 1]) {
          values <- c(values, 1 / (1 - a) * (sup_inf(f1, P, i)[1]) - a * sup_inf(f2, P, i)[2])
        }
      }
    }
```

```
  }
  return(values)
}
```

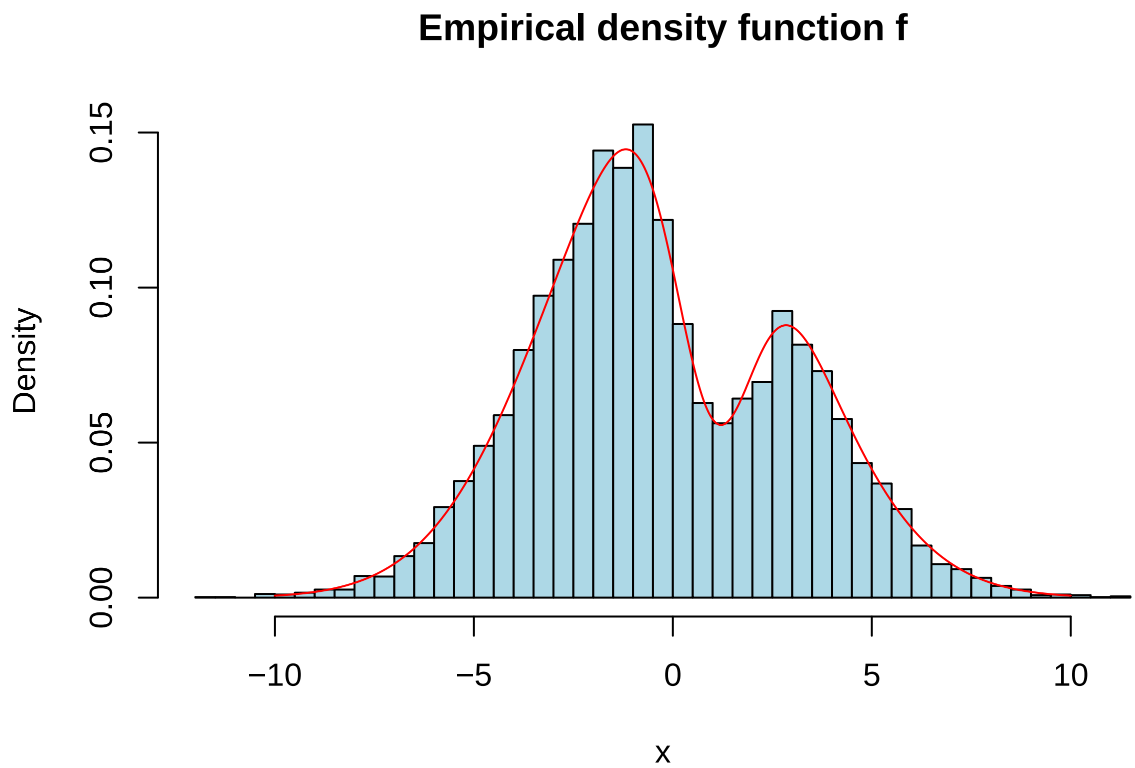
We plot the function f and the dominating function g for different sizes of the partition.

```
library(ggplot2)

X <- seq(-12, 12, length.out = 1000)

# Plot for different k with ggplot on same graph
k_values <- c(10, 20, 50, 100)
P_values <- lapply(k_values, create_partition)
g_values <- lapply(P_values, g, X)

ggplot() +
  geom_line(aes(x = X, y = f(X)), col = "red") +
  geom_line(aes(x = X, y = g(X, P_values[[1]])), col = "green") +
  geom_line(aes(x = X, y = g(X, P_values[[2]])), col = "orange") +
  geom_line(aes(x = X, y = g(X, P_values[[3]])), col = "purple") +
  geom_line(aes(x = X, y = g(X, P_values[[4]])), col = "black") +
  labs(title = "Dominating function g of f for different size of partition", x = "x", y = "Density") +
  theme_minimal()
```



Here, we implement the algorithm of accept-reject with the given partition and an appropriate function g to compute f.

```r
set.seed(123)

g_accept_reject <- function(x, P) {
  if (x < P[1] | x >= P[length(P)]) {
    return(f1(x))
  } else {
    for (i in seq_along(P)) {
      if (x >= P[i] & x < P[i + 1]) {
        return(sup_inf(f1, P, i)[1])
      }
    }
  }
}

stratified <- function(n, P) {
  samples <- numeric(0)
  rate <- 0
  while (length(samples) < n) {
    x <- rnorm(1, mu1, s1)
    u <- runif(1)
    if (u <= f(x) * (1 - a) / g_accept_reject(x, P)) {
      samples <- c(samples, x)
    }
    rate <- rate + 1
  }
  list(samples = samples, acceptance_rate = n / rate)
}

n <- 10000
k <- 100

P <- create_partition(k)
samples <- stratified(n, P)
X <- seq(-10, 10, length.out = 1000)

hist(samples$samples, breaks = 50, freq = FALSE, col = "lightblue", main = "Empirical density function i
lines(X, f(X), col = "red")
```



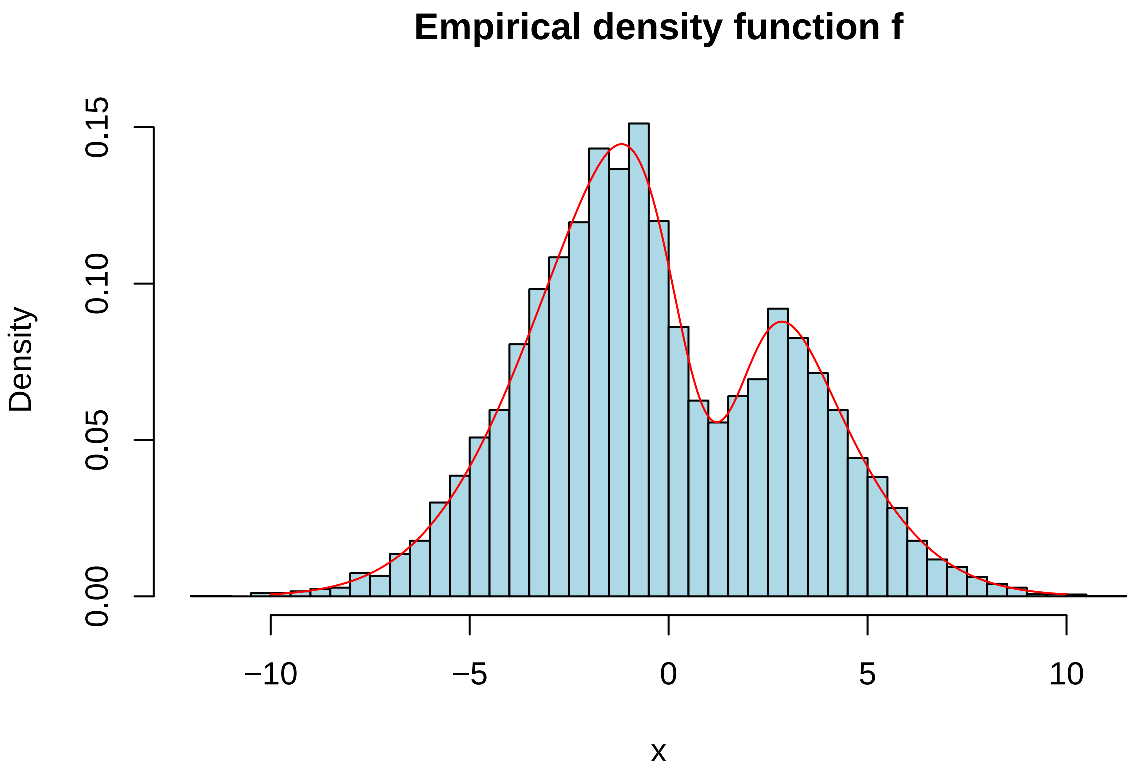
We also compute the acceptance rate of the algorithm.

```
theorical_acceptance_rate <- 1 - a
cat(sprintf("Empirical acceptance rate: %f, Theoretical acceptance rate: %.1f \n", samples$acceptance_ra
```

```
## Empirical acceptance rate: 0.784498, Theoretical acceptance rate: 0.8
```

```
set.seed(123)

stratified_delta <- function(n, delta) {
  samples <- numeric(0)
  P <- create_partition(n * delta)
  rate <- 0
  while (length(samples) < n | rate < delta * n) {
    x <- rnorm(1, mu1, s1)
    u <- runif(1)
    if (u <= f(x) * delta / g_accept_reject(x, P)) {
      samples <- c(samples, x)
    }
    rate <- rate + 1
  }
  list(samples = samples, partition = P, acceptance_rate = n / rate)
}
```

```r
n <- 10000
delta <- 0.8

samples_delta <- stratified_delta(n, delta)
X <- seq(-10, 10, length.out = 1000)

hist(samples_delta$samples, breaks = 50, freq = FALSE, col = "lightblue", main = "Empirical density func
lines(X, f(X), col = "red")
```



Question 12

We also compute the acceptance rate of the algorithm.

```r
theorical_acceptance_rate <- 1 - a
cat(sprintf("Empirical acceptance rate: %f, Theoretical acceptance rate: %f \n", samples_delta$acceptanc
```

```
## Empirical acceptance rate: 0.803213, Theoretical acceptance rate: 0.800000
```

Now, we will test the stratified_delta function for different delta:

```r
set.seed(123)
n <- 1000
deltas <- seq(0.1, 1, by = 0.1)

for (delta in deltas) {
  samples <- stratified_delta(n, delta)
```

```
  cat(sprintf("Delta: %.1f, Empirical acceptance rate: %f \n", delta, samples$acceptance_rate))
}
```

```
## Delta: 0.1, Empirical acceptance rate: 0.097618
## Delta: 0.2, Empirical acceptance rate: 0.199840
## Delta: 0.3, Empirical acceptance rate: 0.304321
## Delta: 0.4, Empirical acceptance rate: 0.392311
## Delta: 0.5, Empirical acceptance rate: 0.503271
## Delta: 0.6, Empirical acceptance rate: 0.594177
## Delta: 0.7, Empirical acceptance rate: 0.692521
## Delta: 0.8, Empirical acceptance rate: 0.786782
## Delta: 0.9, Empirical acceptance rate: 0.848896
## Delta: 1.0, Empirical acceptance rate: 0.858369
```

Cumulative density function.

Question 13 The cumulative density function $\forall x \in \mathbb{R}$ $F_X(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{\mathbb{R}} f(t)h(t), dt$ where $h(x) = \not\Vdash_{X_n \leq x}$

For a given $x \in \mathbb{R}$, a Monte Carlo estimator $F_n(x) = \frac{1}{n} \sum_{i=1}^{n} h(X_i)$ where h is the same function as above and $(X_i)_{i=1}^{n} \sim^{iid} X$

Question 14 As $X_1, \ldots, X_n$ are iid and follows the law of X, and h is continuous and positive, we have $h(X_1), \ldots h(X_n)$ are iid and $\mathbb{E}[h(X_i)] < +\infty$. By the law of large numbers, we have $F_n(X) = \frac{1}{n} \sum_{i=1}^{n} h(X_i) \xrightarrow{a.s} \mathbb{E}[h(X_1)] = F_X(x)$.

Moreover, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \leq N$, $|F_n(x) - F_X(x)| < \epsilon$, ie, $sup_{x \in \mathbb{R}} |F_n(x) - F_X(x)| \xrightarrow{a.s} 0$, by Glivenko-Cantelli theorem.

Hence, F_n is a good estimate of F_X as a function of x.

```
set.seed(123)
n <- 10000
Xn <- (rnorm(n, mu1, s1) - a * rnorm(n, mu2, s2)) / (1 - a)
X <- seq(-10, 10, length.out = n)

h <- function(x, Xn) {
  return(Xn <= x)
}

# Fn empirical
empirical_cdf <- function(x, Xn) {
  return(mean(h(x, Xn)))
}

# F theoretical
F <- function(x) {
  Fx <- pnorm(x, mu1, s1) - a * pnorm(x, mu2, s2)
  return(Fx / (1 - a))
}
```

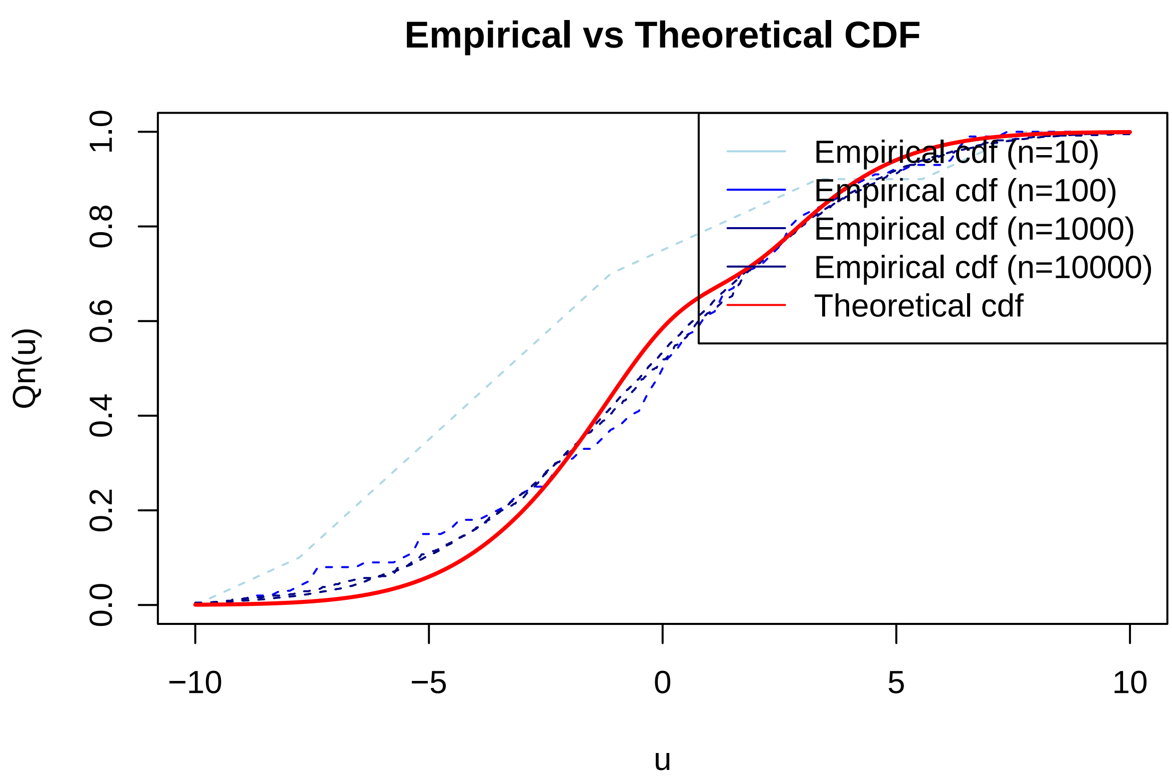
```r
cat(sprintf("Empirical cdf: %f, Theoretical cdf: %f \n", empirical_cdf(X, Xn), mean(F(Xn))))
```

Question 15

```
## Empirical cdf: 0.508300, Theoretical cdf: 0.505263
```

Now we plot the empirical and theoretical cumulative density functions for different n.

```r
n_values <- c(10, 100, 1000, 10000)
colors <- c("lightblue", "blue", "darkblue", "navy")

plot(NULL, xlim = c(-10, 10), ylim = c(0, 1), xlab = "u", ylab = "Qn(u)", main = "Empirical vs Theoreti

for (i in seq_along(n_values)) {
  n <- n_values[i]
  X <- seq(-10, 10, length.out = n)
  Xn <- (rnorm(n, mu1, s1) - a * rnorm(n, mu2, s2)) / (1 - a)
  lines(X, sapply(X, empirical_cdf, Xn = Xn), col = colors[i], lt = 2)
}

lines(X, F(X), col = "red", lty = 1, lw = 2)
legend("topright", legend = c("Empirical cdf (n=10)", "Empirical cdf (n=100)", "Empirical cdf (n=1000)"
```



Question 16 As $X_1, \dots, X_n$ are iid and follows the law of X, and h is continuous and positive, we have $h(X_1), \dots h(X_n)$ are iid and $\mathbb{E}[h(X_i)^2] < +\infty$. By the Central Limit Theorem, we have $\sqrt{n}\frac{(F_n(x)-F_X(x))}{\sigma} \xrightarrow{d} \mathcal{N}(0,1)$ where $\sigma^2 = Var(h(X_1))$.

So we have $\lim_{x\to\infty} \mathbb{P}(\sqrt{n}\frac{(F_n(x)-F_X(x))}{\sigma} \leq q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}) = 1-\alpha$

So by computing the quantile of the normal distribution, we can have a confidence interval for $F_X(x)$: $F_X(x) \in [F_n(x) - \frac{q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}\sigma}{\sqrt{n}}; F_n(x) + \frac{q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)}\sigma}{\sqrt{n}}]$

```
set.seed(123)
Fn <- empirical_cdf(X, Xn)
sigma <- sqrt(Fn - Fn^2)
q <- qnorm(0.975)
interval <- c(Fn - q * sigma / sqrt(n), Fn + q * sigma / sqrt(n))
cat(sprintf("Confidence interval: [%f, %f] \n", interval[1], interval[2]))
```

```
## Confidence interval: [0.501203, 0.520797]
```

```
compute_n_cdf <- function(x, interval_length = 0.01) {
  q <- qnorm(0.975)
  ceiling((q^2 * F(x) * (1 - F(x))) / interval_length^2)
}

x_values <- c(-15, -1)
n_values <- sapply(x_values, compute_n_cdf)

data.frame(x = x_values, n = n_values)
```

Question 17

```
##     x    n
## 1 -15    1
## 2  -1 9530
```

We notice that the size of the sample needed to estimate the cumulative density function is higher for values of x that are close to the mean of the distribution. At $x = -1$, we are on the highest peek of the function and at $x = -15$ we are on the tail of the function.

Empirical quantile function

Question 18 We define the empirical quantile function defined on $(0,1)$ by : $Q_n(u) = inf\{x \in \mathbb{R} : u \leq F_n(x)\}$. We recall the estimator $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} ⊮_{X_i \leq x}$

So we have $Q_n(u) = inf\{x \in \mathbb{R} : u \leq \frac{1}{n}\sum_{i=1}^{n} ⊮_{X_i \leq x}\} = inf\{x \in \mathbb{R} : n.u \leq \sum_{i=1}^{n} ⊮_{X_i \leq x}\}$

We sort the sample $(X_1, \dots, X_n)$ in increasing order, and we define $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ the order statistics of the sample.

As $\sum_{i=1}^{n} ⊮_{X_{(i)} \leq x} = k$ where $X_{(k)} = max\{i = 1, \dots, n; X_{(i)} \leq x\}$

But as F_n is a step function, we can simplify the expression to $Q_n(u) = X_{(k)}$ where $k = \lceil n \cdot u \rceil$ and $X_{(k)}$ is the k-th order statistic of the sample $(X_1, \dots, X_n)$.

Question 19 We note $Y_{j,n} := \nVdash_{X_{n,j} < Q(u) + \frac{t}{\sqrt{n}} \frac{\sqrt{u(1-u)}}{f(Q(u))}}$

We know that $(X_{n,j})$ is iid as X is bounded in j and n.

Let $\Delta_n = \frac{t}{\sqrt{n}} \frac{\sqrt{u(1-u)}}{f(Q(u))}$. We have $F_n(X) = \frac{1}{n} \sum_{j=1}^{n} \nVdash_{X_{n,j}} < Q(u) + \Delta_n$

then $\frac{1}{n} \sum_{j=1}^{n} Y_{j,n} = F_n(Q(u) + \Delta_n)$ by definition of the empirical quantile $F_n(Q_n(u)) = u$

By Taylor formula, we got $F_n(Q(u) + \Delta_n) = F_n(Q(u)) + \Delta_n f(Q(u))$

By Lindbergh-Levy Central Limit Theorem, applied to $F_n(Q(u))$ as $\mathbb{E}[F_n(Q(u))] = u < +\infty$ and $Var(F_n(Q(u))) = u(1-u) < +\infty$, we have $\frac{\sqrt{n}(F_n(Q(u)) - u)}{\sqrt{u(1-u)}} \to \mathcal{N}(0, 1)$

then $F_n(Q(u)) = u + \frac{1}{\sqrt{n}} Z$ with $Z \sim \mathcal{N}(0, u(1-u))$

Thus $F_n(Q(u) + \Delta_n) = u + \frac{1}{\sqrt{n}} Z + \Delta_n f(Q(u))$

By substituting $Q_n(u) = Q(u) + \Delta_n$, we have

$$F_n(Q_n(u)) = F_n(Q(u) + \Delta_n) \Leftrightarrow u = u + \frac{1}{\sqrt{n}} Z + \Delta_n f(Q(u)) \Leftrightarrow \Delta_n = -\frac{1}{\sqrt{n}} \frac{Z}{f(Q(u))}$$

As $Q_n(u) = Q(u) + \Delta_n \Rightarrow \Delta_n = Q_n(u) - Q(u)$

Then we have

$$Q_n(u) - Q(u) = -\frac{1}{\sqrt{n}} \frac{Z}{f(Q(u))} \Leftrightarrow \sqrt{n}(Q_n(u) - Q(u)) = \frac{Z}{f(Q(u))} \Leftrightarrow \sqrt{n}(Q\_n(u) - Q(u)) \sim \mathcal{N}(0, \frac{u(1-u)}{f(Q(u))^2})$$

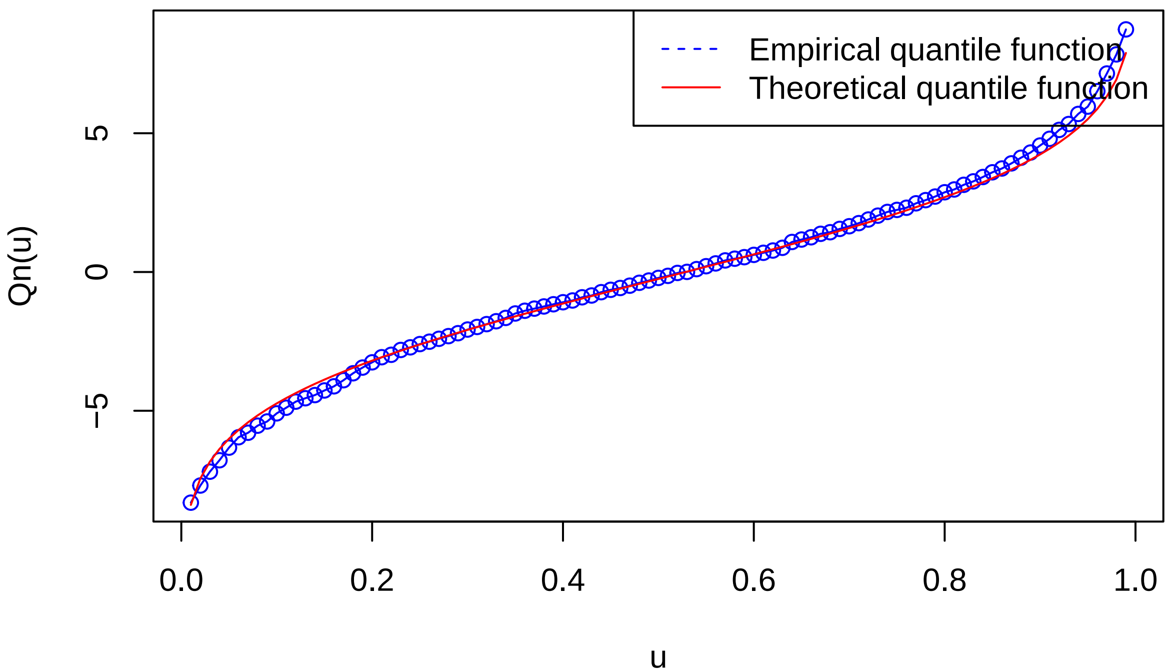
Question 20 When $u \to 0$, $Q(u)$ corresponds to the lower tail of the distribution, and when $u \to 1$, $Q(u)$ corresponds to the upper tail of the distribution.

So $f(Q(u))$ is higher when u is close to 0 and close to 1, so the variance of the estimator is higher for values of u that are close to 0 and 1. So we need a higher sample size to estimate the quantile function for values of u that are close to 0 and 1.

```r
set.seed(123)

empirical_quantile <- function(u, Xn) {
  sorted_Xn <- sort(Xn)
  k <- ceiling(u * length(Xn))
  sorted_Xn[k]
}
```

```r
set.seed(123)
n <- 1000
Xn <- (rnorm(n, mu1, s1) - a * rnorm(n, mu2, s2)) / (1 - a)
u_values <- seq(0.01, 0.99, by = 0.01)
Qn_values <- sapply(u_values, empirical_quantile, Xn = Xn)
```

```
plot(u_values, Qn_values, col = "blue", xlab = "u", ylab = "Qn(u)", type = "o")
lines(u_values, (qnorm(u_values, mu1, s1) - a * qnorm(u_values, mu2, s2)) / (1 - a), col = "red")
legend("topright", legend = c("Empirical quantile function", "Theoretical quantile function"), col = c(
```



Question 21

Question 22 We can compute the confidence interval of the empirical quantile function using the Central Limit Theorem.

We obtain the following formula for the confidence interval of the empirical quantile function:

$Q(u) \in [Q_n(u) - q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} \frac{\sqrt{u(1-u)}}{\sqrt{n} f(Q(u))}; Q_n(u) + q_{1-\frac{\alpha}{2}}^{\mathcal{N}(0,1)} \frac{\sqrt{u(1-u)}}{\sqrt{n} f(Q(u))}]$

```
f_q <- function(u) {
  f(1 / (1 - a) * (qnorm(u, mu1, s1) - a * qnorm(u, mu2, s2)))
}

compute_n_quantile <- function(u, interval_length = 0.01) {
  q <- qnorm(0.975)
  ceiling((q^2 * u * (1 - u)) / (interval_length^2 * f_q(u)^2))
}

u_values <- c(0.5, 0.9, 0.99, 0.999, 0.9999)
n_values <- sapply(u_values, compute_n_quantile)

data.frame(u = u_values, n = n_values)
```

```
##        u           n
## 1 0.5000      667059
## 2 0.9000      934418
## 3 0.9900    13944215
## 4 0.9990   338588623
## 5 0.9999 10183256354
```

We deduce that the size of the sample needed to estimate the quantile function is higher for values of u that are close to 1. This corresponds to the deduction made in question 20.